Application of nonlinear deformation algebra to a physical system with Pöschl-Teller potential

1998 J. Phys. A: Math. Gen. 31 6473

(http://iopscience.iop.org/0305-4470/31/30/012)

J. Phys. A: Math. Gen. **31** (1998) 6473–6481. Printed in the UK PII: S0305-4470(98)89358-X

Application of nonlinear deformation algebra to a physical system with Pöschl–Teller potential

Jing-Ling Chen, Yong Liu and Mo-Lin Ge

Theoretical Physics Division, Nankai Institute of Mathematics, Nankai University, Tianjin 300071, People's Republic of China†

Received 18 November 1997

Abstract. Nonlinear deformation algebra are realized in a physical system with Pöschl–Teller potential. The raising and lowering operators satisfying this algebra are constructed, from which the eigenproblem of the system can be exactly solved by the operator method. The physical meaning of two deforming functions involved in this algebra is also found. In addition, $SU(1,1)$ algebra is obtained naturally, and discussions on the coherent state are also made.

1. Introduction

For a long time, all efforts to develop the theory of symmetry in physics were restricted to the *linear* case, i.e. to Lie groups and Lie algebras which are among the cornerstones of modern physics. However, it has been stressed that there was no *physical* reason for symmetries to be linear and that Lie group theory was therefore too restrictive [1]. Since the discovery of quantum algebra (q-deformation of Lie algebras) by Drinfeld [2] as a type of nonlinear algebra setting for the inverse scattering problem, this algebra has found many applications in diverse domains of physics, such as two-dimensional (2D) integrable models, systems on lattices and 2D conformal field theories [3, 4]. Recently, nonlinear deformations of $SU(2)$ and $SU(1,1)$ algebras (NLDA) with two deforming functions $g(J_0)$ and $f(J_0)$ were introduced by Delbecq and Quesne [5]. For the case $g(J_0) = 1$ with arbitrary $f(J_0)$, the algebra has been studied by Rocek who has represented the theory and suggested that the presence of an arbitrary function might prove useful in some applications to physical models [6]. Obviously, it includes $SU(2)$, $SU(1,1)$ and $SU_q(2)$ etc as special cases [7–9] according to different choices of the two deforming functions. Although the probability of physical applications is suggested, this kind of nonlinear algebra has still limited itself to mathematical discussions and has not yet been used as a powerful technique in physics. In order to fully explore the roles played by the *nonlinear algebras* in physics further, we show that nonlinear deformations of $SU(2)$ and $SU(1,1)$ algebras can be realized in a physical system with Pöschl–Teller potential, which is one of the exactly solvable one-dimensional quantum-mechanical potentials. Due to this algebra, the eigenproblem of the system can be determined by the operator method without dealing with the Schrödinger equation.

This work is organized as follows: in section 2, we review the nonlinear deformation algebra and give definitions of the raising and lowering operators together with the Hamiltonian operator, which form generators of NLDA. In section 3 the energy spectrum

† E-mail address: yuantuo@sun.nankai.edu.cn

and eigenkets of the system are obtained by the operator method. We point out the physical meaning of two deforming functions and obtain the $SU(1,1)$ algebra from NLDA naturally in section 4. Some conclusions are discussed in the last section.

2. NLDA realized in a physical system with Pöschl–Teller potential

As noted in [5], the 'nonlinear deformation algebras' generated by the three operators $J_0 = (J_0)^\dagger$, J_+ and $J_- = (J_+)^\dagger$, satisfy the following commutation relations

$$[J_0, J_-] = -J_- g(J_0) \qquad [J_0, J_+] = g(J_0) J_+ \qquad [J_-, J_+] = f(J_0) \tag{1}$$

and the algebras have a Casimir operator of the type

$$C = J_- J_+ + h(J_0) \tag{2}$$

where $g(J_0)$, $f(J_0)$ and $h(J_0)$ are three real 'deforming functions' of J_0, holomorphic in the neighbourhood of zero and satisfy the following relation

$$h(J_0) - h(J_0 - g(J_0)) = f(J_0). \tag{3}$$

Now, we consider a physical system with a particle of mass m moving in the symmetric Pöschl–Teller potential, which takes the form

$$V(x) = \frac{V_0}{\cos^2(kx)} \tag{4}$$

where V_0 is a constant and k is a parameter. The corresponding Hamiltonian reads

$$H = \frac{p^2}{2m} + V(x). \tag{5}$$

Assuming $|\psi\rangle$ and E are eigenfunction and eigenenergy of the system, respectively, we need to solve the eigenvalue problem

$$H|\psi\rangle = E|\psi\rangle. \tag{6}$$

As is well known, with the Pöschl–Teller potential this eigenvalue problem can be exactly solved by dealing with the Schrödinger equation directly; the eigenfunctions are represented by hypergeometric polynomials and trigonometric functions [10]. In the following, we shall construct the raising and lowering operators for the Hamiltonian system and solve this problem by the operator method. In fact, the operators we construct together with the Hamiltonian satisfy the nonlinear deformation algebra introduced in (1). The results from solving the Schrödinger equation can be used to test the validity of those from the operator method.

We start from the following definitions

$$X = \sin(kx) \qquad P = \tfrac{1}{2} k \{\cos(kx), p\} \tag{7}$$

where the symbol { , } represents the anti-commutator. Using the fundamental commutation relation $[x, p] = \mathrm{i}\hbar$, we obtain

$$[X, P] = \mathrm{i}\hbar k^2 (1 - X^2) \qquad [H, X] = -\frac{\mathrm{i}\hbar}{m} P \qquad [H, P] = \mathrm{i}\hbar k^2 \left(HX + XH - \frac{1}{2}\epsilon X \right) \tag{8}$$

with $\epsilon = (\hbar^2 k^2 / 2m)$.

The last two equations of (8) can be rewritten in the following matrix form

$$H(X, P) = (X, P) G \tag{9}$$

where G is a (2×2) matrix

$$G = \begin{pmatrix} H & \mathrm{i}\hbar k^2 \left(2H - \frac{1}{2}\epsilon\right) \\ -\mathrm{i}\hbar/m & H + 2\epsilon \end{pmatrix}. \tag{10}$$

Solving the equation

$$\det(G - \lambda I) = 0 \tag{11}$$

we obtain eigenvalues of G as follows

$$\lambda_1 = H + \epsilon + 2\sqrt{\epsilon H} = (\sqrt{H} + \sqrt{\epsilon})^2 \qquad \lambda_2 = H + \epsilon - 2\sqrt{\epsilon H} = (\sqrt{H} - \sqrt{\epsilon})^2. \tag{12}$$

The diagonalized G can be written in the form

$$G = S\Lambda S^{-1} \tag{13}$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{14}$$

and the diagonalizing matrix

$$S = \begin{pmatrix} \epsilon - 2\sqrt{\epsilon H} & \epsilon + 2\sqrt{\epsilon H} \\ \mathrm{i}\hbar/m & \mathrm{i}\hbar/m \end{pmatrix}. \tag{15}$$

Let $F(H)$ be a real function of H, holomorphic in the neighbourhood of zero, and from (9) we have the following operator equation

$$F(H)(X, P) = (X, P)SF(\Lambda)S^{-1}. \tag{16}$$

By defining

$$g(H) = H - \lambda_2 = -\epsilon + 2\sqrt{\epsilon H} \qquad f(H) = 4\gamma^2\epsilon(\lambda_1 - H) = 4\gamma^2\epsilon^2\left(1 + 2\sqrt{\frac{H}{\epsilon}}\right)$$
$$b = \gamma\left[X(\lambda_1 - H) + \frac{\mathrm{i}\hbar}{m}P\right] \qquad b^+ = (b)^\dagger = \gamma\left[(\lambda_1 - H)X - \frac{\mathrm{i}\hbar}{m}P\right] \tag{17}$$

where γ is a changeable parameter which enables b and b^+ to be dimensionless, from equations (16) and (17), we get

$$[H, b] = -bg(H) \qquad [H, b^+] = g(H)b^+ \qquad [b, b^+] = f(H). \tag{18}$$

Due to relation (3), we find

$$h(H) = 4\gamma^2\epsilon\lambda_1 \tag{19}$$

so that according to (2) the Casimir operator can also be determined.

Hence, we have introduced the raising operator b^+ and the lowering operator b in (17) and successfully embedded the operators b, b^+ and H into (18) which is the nonlinear deformation algebra introduced in (1). Thus, we show that NLDA can be realized in the symmetric Pöschl–Teller model. In the next section we show that the eigenvalue problem (6) can be solved by the operator method formed by b and b^+.

3. Determination of eigenenergies and eigenkets of the system

From equations (16) and (17), we obtain

$$F(H)b = bF(H - g(H)) \qquad F(H)b^+ = b^+F(H + 2\epsilon + g(H)) \tag{20}$$

and

$$b^+ = -\gamma\left[X(\epsilon - 2\sqrt{\epsilon H}) + \frac{\mathrm{i}\hbar}{m}P\right]\left(\frac{\epsilon}{\sqrt{\epsilon H}} + 1\right). \tag{21}$$

The eigenequation is

$$H|\psi_n\rangle = E_n|\psi_n\rangle. \tag{22}$$

From the first equation of (18) we have

$$[H, b]|\psi_n\rangle = -bg(H)|\psi_n\rangle \tag{23}$$

or

$$H(b|\psi_n\rangle) = (E_n - g(E_n))(b|\psi_n\rangle). \tag{24}$$

Let $|\psi_{n-1}\rangle = b|\psi_n\rangle \neq 0$ denote a new eigenket of H with the eigenvalue

$$E_{n-1} = E_n - g(E_n) \tag{25}$$

or

$$g(E_n) = E_n - E_{n-1} \tag{26}$$

then from (25) we easily obtain

$$\sqrt{E_n} = \sqrt{E_{n-1}} + \sqrt{\epsilon} \tag{27}$$

so that

$$\sqrt{E_n} = \sqrt{E_0} + n\sqrt{\epsilon} \tag{28}$$

or

$$E_n = \epsilon(n+\nu)^2 \qquad n = 0, 1, 2, \ldots \tag{29}$$

with

$$\nu = \sqrt{\frac{E_0}{\epsilon}} \tag{30}$$

where E_0 is the zero-point energy, i.e. the lowest eigenvalue, which will be determined later. To make it clearer and to determine the eigenkets, we take the following step. Let $|\psi_0\rangle$ be the ground state, so we have

$$H|\psi_0\rangle = E_0|\psi_0\rangle \tag{31}$$

and

$$b|\psi_0\rangle = 0. \tag{32}$$

By solving these equations, one can obtain

$$|\psi_0\rangle = \cos^{\nu}(kx). \tag{33}$$

After substituting it into (31), we find that

$$V_0 = \frac{\hbar^2k^2}{2m}\nu(\nu - 1) \tag{34}$$

so that

$$\nu = \frac{1}{2}\left(1 - \sqrt{1 + \frac{4V_0}{\epsilon}}\right). \tag{35}$$

Combining (29) with (35) we therefore find that the energy spectrum of a particle moving in Pöschl–Teller potential is given by

$$E_n = \frac{\hbar^2 k^2}{2m}\left[n + \frac{1}{2}\left(1 - \sqrt{1 + \frac{8mV_0}{\hbar^2 k^2}}\right)\right]^2 \qquad n = 0, 1, 2, \ldots. \tag{36}$$

The wavefunction $|\psi_{n+1}\rangle$ can be constructed by the action of b^+ on $|\psi_n\rangle$

$$\begin{aligned} |\psi_{n+1}\rangle &= b^+|\psi_n\rangle \\ &= -\gamma\left[X\epsilon(1 - 2(n+\nu) + \frac{i\hbar}{m}P\right]\left(\frac{1}{n+\nu} + 1\right)|\psi_n\rangle \\ &\propto \left[(n+\nu)\sin(kx) - \frac{1}{k}\cos(kx)\frac{\mathrm{d}}{\mathrm{d}x}\right]|\psi_n\rangle. \end{aligned} \tag{37}$$

In more detail we write the first few (unnormalized) as follows:

$$\begin{aligned} |\psi_1\rangle &= \cos^\nu(kx)\sin(kx) \\ |\psi_2\rangle &= \cos^\nu(kx)[2(1+\nu)\sin^2(kx) - 1] \\ |\psi_3\rangle &= \cos^\nu(kx)\sin(kx)[2(2+\nu)\sin^2(kx) - 3] \\ |\psi_4\rangle &= \cos^\nu(kx)[4(2+\nu)(3+\nu)\sin^4(kx) - 12(2+\nu)\sin^2(kx) + 3] \\ &\text{etc} \end{aligned} \tag{38}$$

and rewrite these eigenfunctions by hypergeometric polynomials. For even values of n, we have

$$|\psi_n\rangle = C_n \cos^\nu(kx)\,{}_2F_1\left(-\frac{n}{2}, \frac{n}{2} + \nu; \frac{1}{2}; \sin^2(kx)\right) \tag{39}$$

while for odd n it corresponds to

$$|\psi_n\rangle = C_n \cos^\nu(kx)\sin(kx)\,{}_2F_1\left(-\frac{n}{2} + \frac{1}{2}, \frac{n}{2} + \nu + \frac{1}{2}; \frac{3}{2}; \sin^2(kx)\right) \tag{40}$$

where C_n is the normalizing constant.

Thus, we have obtained the energy spectrum and wavefunctions for a particle moving in the Pöschl–Teller potential by the operator method. Obviously they are coincident with those obtained from the solution of the Schrödinger equation [10]. These eigensolutions and eigenfunctions given in the analytic, special-function (no hypergeometric function) form including analytic normalizations can be seen in [11].

It is also interesting to consider the limiting case, where $V_0 \to 0$, the problem becomes one of a particle in an infinite square well. Taking $k = \pi/L$, where L is the width of the potential well, the quantity ν tends to zero and, as we expect, the energy (36) reduces to

$$E_n = \frac{\hbar^2}{2m}\frac{\pi^2 n^2}{L^2} \qquad n = 1, 2, \ldots. \tag{41}$$

4. Physical meaning of two deforming functions and $SU(1,1)$ algebra

From equation (18) one can easily prove that the two arbitrary deforming functions $g(J_0)$ and $f(J_0)$ can be rewritten into the functions of H, namely $g(H)$ and $f(H)$. Equation (26) shows that $g(E_n)$ is an interval between two adjacent energy levels specified by degrees n and $n-1$, respectively. Since $g(E_n) = -\epsilon + 2\sqrt{\epsilon E_n}$ depends on E_n, the energy intervals of the system are unequal. In order to clearly see the physical meaning of $g(H)$, we can refer to the one-dimensional linear harmonic oscillator, whose Hamiltonian H, raising and lowering operators a^+ and a satisfy the following algebra

$$[H, a] = -\hbar\omega a \qquad [H, a^+] = \hbar\omega a^+ \qquad [a, a^+] = 1. \tag{42}$$

Obviously they are also a special case of NLDA in the following meaning

$$g(H) = \hbar\omega \qquad f(H) = 1. \tag{43}$$

So $g(H)$ is a constant and does not depend on H. As we know, the energy spectrum of a linear harmonic oscillator is discrete and its energy levers are equally spaced by the interval $g(H) = \hbar\omega$. Based on this, the physical meaning of $g(H)$ is clear, namely, it is just an *energy interval operator* giving the interval between two adjacent energy levers when it acts on a certain eigenfunction $|\psi_n\rangle$.

From the last equation of (18), one finds that

$$H = f^{-1}([b, b^+]) \tag{44}$$

where f^{-1} is the inverse function of f, equation (44) means that the Hamiltonian operator H can be represented by the raising and lowering operators, which is just the physical meaning of $f(H)$.

If we select

$$4\gamma^2\epsilon^2 = 1 \tag{45}$$

and use the second equation of (17), then we obtain

$$H = \frac{\hbar^2 k^2}{8m}(bb^+ - b^+b - 1)^2. \tag{46}$$

Finally, we see that, corresponding to equation (20) an equivalent equation can be obtained as follows

$$F(\sqrt{H})b = bF(\sqrt{H} - \sqrt{\epsilon}) \qquad F(\sqrt{H})b^+ = b^+F(\sqrt{H} + \sqrt{\epsilon}). \tag{47}$$

Thus, if we define

$$J_0 = \frac{1}{2} + \sqrt{\frac{H}{\epsilon}} \qquad J_+ = b^+ \qquad J_- = b \tag{48}$$

combine these with (45) and let $F(\sqrt{H}) = \sqrt{H}$ in (47), it is easy to verify that

$$[J_0, J_+] = J_+ \qquad [J_0, J_-] = -J_- \qquad [J_+, J_-] = -2J_0. \tag{49}$$

So we can obtain here a simpler algebra $SU(1,1)$ from NLDA naturally. The $SU(1,1)$ symmetry for the Pöschl–Teller potential or other potentials has been discussed in many literatures [12–14]. However, in our work we achieve it from the point of view of the nonlinear deformation algebra.

5. Discussion and conclusions

(1) To the $SU(1,1)$ algebra realized in equations (48) and (49), we have the Hermiticity properties

$$(J_0)^\dagger = J_0 \qquad (J_+)^\dagger = J_- \tag{50}$$

which are requirements in the definition of NLDA (see equation (1)).

Explicit forms of the lowering and raising operators are

$$\begin{aligned} b &= \sin(kx)\sqrt{\frac{H}{\epsilon}} + \frac{\mathrm{i}}{\hbar k}\cos(kx)p \\ b^+ &= (b)^\dagger = \sqrt{\frac{H}{\epsilon}}\sin(kx) - \frac{\mathrm{i}}{\hbar k}p\cos(kx) \\ p &= -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}. \end{aligned} \tag{51}$$

Note that they are the strict operators associated with the Hamiltonian operator H. After equations (21) and (51) acting on $|\psi_n\rangle$, one can obtain the lowering and raising operators which are n-dependent for the Pöschl–Teller system

$$\begin{aligned} b_n &= \sin(kx)(n+\nu) + \frac{\mathrm{i}}{\hbar k}\cos(kx)p \\ b_n^+ &= \sin(kx)(n+\nu) - \frac{\mathrm{i}}{\hbar k}\cos(kx)p. \end{aligned} \tag{52}$$

These kind of operators have been used in [15] to show the coherent state for the Pöschl–Teller system. However, b_n and b_n^+ are not mutually adjoint.

The Hermitian operators X and P are 'natural quantum variables' as referred to in [15]. It has also been shown that they can be written as the sum and difference of $[b_n + (b_n^+)^\dagger]$ and $[(b_n)^\dagger + b_n^+]$, which are adjoints of each other. Specifically

$$X = \frac{1}{4(n+\nu)}\{[b_n + (b_n^+)^\dagger] + [(b_n)^\dagger + b_n^+]\} \qquad P = \frac{\hbar k^2}{4\mathrm{i}}\{[b_n + (b_n^+)^\dagger] - [(b_n)^\dagger + b_n^+]\}. \tag{53}$$

However, from (21) and the third equation of (17) we have

$$(b, b^+) = (X, P)M \tag{54}$$

with M a matrix depending on the parameter H

$$M = \gamma \begin{pmatrix} \lambda_1 - H & -(\lambda_2 - H)\left(\sqrt{\frac{\epsilon}{H}} + 1\right) \\ \frac{\mathrm{i}\hbar}{m} & -\frac{\mathrm{i}\hbar}{m}\left(\sqrt{\frac{\epsilon}{H}} + 1\right) \end{pmatrix} \tag{55}$$

so that

$$(X, P) = (b, b^+)M^{-1} \tag{56}$$

where M^{-1} is the inverse matrix of M. Consequently, the 'natural quantum variables' can be expressed in terms of the strict operators b, b^+ and H.

(2) Eventually, we want to simply mention the coherent state for the Pöschl–Teller system, since we have established the $SU(1,1)$ algebra in it. The term *coherent state* (CS) was first coined by Glauber in 1963, who constructed the eigenstates of the annihilation operator of the quantum harmonic oscillator in order to study the electromagnetic correlation functions, a subject of great importance in quantum optics [16]. Extensive applications of

such a state can be seen in [17] and references therein. Three kinds of standard definitions for the Weyl–Heisenberg algebra CS result in the same state in the simple harmonic-oscillator system, but for other algebras the equivalents will not always hold [18–21]. The coherent states were generalized in different ways by different authors, for instance, Nieto and Simmons have defined the generalized CS as states which satisfied the classical equations of motion; Radcliffe and Perelemov defined them as states displaced from the ground state or a reference state [22–23], while Barut and Girardello [24] developed that the 'new coherent state' are eigenstates of the lowering operator of non-compact groups such as $SU(1, 1)$.

Following the spirit of [22–24], the CS $|\alpha\rangle$ is defined by

$$b|\alpha\rangle = \alpha|\alpha\rangle \qquad D(\alpha)|0\rangle = |\alpha\rangle. \tag{57}$$

As shown in [15], because the energy levels of the Pöschl–Teller system are not equally spaced, then (a) the CS $|\alpha\rangle$ defined earlier is not the same as the minimum-uncertainty coherent state, and (b) the appropriate displacement operator $D(\alpha)$ is not in general an exponential, but can be a more complicated functional. In [18], the $SU(1, 1)$ generalized CS $|\xi, j\rangle$ is defined by

$$|\xi, j\rangle = S(\alpha)|0, j\rangle \qquad J_-|0, j\rangle = 0 \qquad S(\alpha) = \exp(\alpha J_+ - \alpha^* J_-) \tag{58}$$

due to the $SU(1, 1)$ disentangling theorem [25]; an explicit expression for $|\xi, j\rangle$ can be found.

Therefore, for $SU(1, 1)$, different definitions lead to distinct states. In [20, 21], authors introduce the concept of algebra eigenstates (AES) which are defined for an arbitrary Lie group as eigenstates of the elements of the corresponding complex Lie algebra. This concept unifies different definitions of coherent states associated with a dynamical symmetry group which in detail, readers can investigate in the literature.

In summary, we have developed an operator method to exactly solve the physical system with Pöschl–Teller potential and give its energy spectrum. The operators we constructed form a nonlinear deformation algebra as introduced by Delbecq *et al* (1993). The two deforming functions involved in this algebra have a definite physical meaning. In a sense, we successfully realized an application of NLDA to a physical system. Moreover, coherent states are also discussed.

Acknowledgments

This work was partially supported by the National Natural Science Foundation of China. One of the authors (J L Chen) is grateful to the referees for their kind suggestions.

References

[1] de Boer J, Harmsze F and Tjin T 1996 *Phys. Rep.* **272** 139
[2] Drinfeld V G 1988 *Proc. Int. Congr. Math.* (Berkeley, CA: MSRI) 798
Drinfeld V G 1988 *Sov. Math. Dokl.* **36** 212
[3] de Vega H J 1989 *Int. J. Mod. Phys.* A **4** 2371
[4] Alvarez-Gaume L, Gomez C and Sierra G 1989 *Nucl. Phys.* B **319** 155
[5] Delbecq C and Quesne C 1993 *J. Phys. A: Math. Gen.* **26** L127–34
[6] Rocek M 1991 *Phys. Lett.* **255B** 554–7
[7] Sklyanin E K 1982 *Funct. Anal. Appl.* **16** 262
[8] Kulish P P and Reshetikhin N Y 1983 *J. Sov. Math.* **23** 2435
[9] Granovskii Ya I, Zhedanov A S and Grakhovskaya O B 1992 *Phys. Lett.* **278B** 85–8
[10] ter Haar D 1975 *Problems in Quantum Mechanics* (London: Pion)

[11] Nieto M M 1978 *Phys. Rev.* A **17** 1273
[12] Alhassid Y, Gürsey F and Iachello F 1983 *Ann. Phys.* **148** 346
Alhassid Y, Gürsey F and Iachello F 1986 *Ann. Phys.* **167** 181
[13] Eleonsky V M and Korolev V G 1996 *J. Phys. A: Math. Gen.* **29** L241–8
[14] Zeng G J, Zhou S L, Ao S M and Jiang F S 1997 *J. Phys. A: Math. Gen.* **30** 1775–83
[15] Nieto M M and Simmons L M Jr 1978 *Phys. Rev. Lett.* **41** 207
Nieto M M and Simmons L M Jr 1979 *Phys. Rev.* D **20** 1321–51
[16] Glauber R J 1963 *Phys. Rev. Lett.* **10** 277
Glauber R J 1963 *Phys. Rev.* **130** 2529
Glauber R J 1963 *Phys. Rev.* **131** 2766
[17] Zhang W M, Feng D H and Gilmore R 1990 *Rev. Mod. Phys.* **62** 867
[18] Buzek V 1990 *J. Mod. Opt.* **37** 303–16
[19] Trifonov D A 1994 *J. Math. Phys.* **35** 2297
[20] Brif C, Vourdas A and Mann A 1996 *J. Phys. A: Math. Gen.* **29** 5873
[21] Brif C 1996 *Ann. Phys.* **251** 180
Brif C 1997 *Int. J. Phys.* **36** 1651
[22] Radcliffe J M 1971 *J. Phys. A: Math. Gen.* **4** 313
[23] Perelemov A M 1972 *Commun. Math. Phys.* **26** 222
[24] Barut A O and Girardello L 1971 *Commun. Math. Phys.* **21** 41
[25] Wodkiewicz K and Eberly J H 1985 *J. Opt. Soc. Am.* B **2** 458